ON THE OPTIMIZATION OF TRACKING SYSTEMS

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Let the equation

$$L_n(y) \equiv a_0(t)\,y^{(n)} + a_1(t)\,y^{(n-1)} + \ldots + a_n(t)\,y = f(t) + c(t)\,f'(t) \qquad (1)$$

with initial conditions

$$y(0) = y'(0) = \ldots y^{(n-1)}(0) = f(0) = 0 \qquad (2)$$

describe a tracking system whose input is supplied not only with a control function $f(t)$ but also with its derivative $f'(t)$, amplified by a variable amplification factor $c(t)$. for the purpose of improving its performance. It is assumed that the function $f(t)$ arriving at the input of the tracking system has been filtered free of high-frequency noise and interference. Nothing is known about $f(t)$ except that

$$|f'(t)| \leqslant m \qquad (3)$$

and $f'(t)$ has a finite number of points of discontinuity in a finite interval of time.

The function $c(t)$ must satisfy the constraint

$$|c(t)| \leqslant M \qquad (4)$$

The error signal $y(t) - f(t)$ will be denoted by $\delta(t, f(\tau), c(\tau))$.

Let us assume that at a fixed instant of time T the modulus of the error signal must not exceed a given value for any value of $f'(t)$ satisfying (3), that is:

$$\sup_f |\delta(T, f(t), c(t))| \leqslant A, \qquad |f'(t)| \leqslant m, \qquad t \in [0, T] \qquad (5)$$

and let the inequality

$$\inf_c \sup_f |\delta(T, f(t), c(t))| = A^\circ < A$$
$$|c(t)| \leqslant M, \qquad |f'(t)| \leqslant m, \qquad t \in [0, T] \tag{6}$$

be satisfied.

We shall now formulate the problem. In the set of integrable functions $c(t)$ satisfying the conditions (4) and (5), required to find the function $c_{\min}(t)$ for which

$$\inf_c \sup_f |\delta_T'(T, f(t), c(t))| \qquad |f'(t)| \leqslant m, \qquad t \in [0, T]$$

will be true.

At time T the values of $y(T)$ and $f(T)$ agree to the best first-order approximation possible for the given conditions for any $f(t)$ satisfying (3).

The problem considered here is directly related to Bulgakov's method of accumulated perturbations [1], since all that is known about $f(t)$ is the condition (3).

It should be noted that the method of supplying a linear combination of the control function $f(t)$ and its derivative is fairly widespread in practice.

In a number of cases the control function may be differentiated exactly in practice, for example, when $f'(t)$ is obtained by means of a tachometer. In other cases, for example, in the case of differentiation by means of RC networks, the error in finding the derivative consists in the fact that instead of the function $f'(t)$ one obtains the function $s(t)$, which is a solution of the equation

$$T^* \frac{dS}{dt} + S = \frac{df}{dt}$$

For sufficiently small values of the time constant T^*, $S(t) \approx f'(t)$. In the present note it is assumed that the differentiation is exact. However, the statement of the problem remains meaningful even for the case when $S(t)$ is supplied to the input of the system instead of $f'(t)$.

We shall assume in what follows that in the interval $[0, T_0]$, where $T_0 > T$, all coefficients $a_i(t)$ of Equation (1) are functions which have $n - i$ continuous derivatives and $a_0(t)$ does not vanish in $[0, T_0]$. These conditions facilitate the calculation of certain functions introduced below. Representing $f(t)$ in the form

$$f(t)=\int_0^t f'(\tau)\,d\tau$$

and changing the order of integration in the resulting repeated integral, taking Equation (2) into account, the error signal $\delta(T, f, c)$ may be represented in the form

$$\delta(T,f,c)=\int_0^T [K_0(T,\tau)+c(\tau)K(T,\tau)]\,f'(\tau)\,d\tau$$

$$K_0(T,\tau)=\int_\tau^T K(T,u)\,du-1, \qquad K(T,\tau)=\sum_{r=1}^n y_r(T)\frac{W_r(\tau)}{a_0(\tau)\,W(\tau)}$$

Here $y_1(t)$, $y_2(t)$, ..., $y_n(t)$ form a fundamental system of solutions of the homogeneous equation corresponding to Equation (1), $W(\tau)$ is the Wronskian of this system, and $W_r(\tau)$ is the cofactor of the element of the last row and rth column of the Wronskian. Therefore

$$K(T,T)=0, \qquad K_0(T,T)=-1$$

Let us recall that the functions $Z_r(\tau) = W_r(\tau)/a_0(\tau)W(\tau)$ form a fundamental system of solutions of the equation

$$M_n(Z)\equiv(-1)^n(a_0Z)^{(n)}+(-1)^{n-1}(a_1Z)^{(n-1)}+\ldots-(a_{n-1}Z)'+a_nZ=0 \qquad (7)$$

Here M_n is the operator conjugate to L_n.

Since $c(\tau)$ and $f'(\tau)$ are integrable, $K(T, \tau)$, $K_0(T, \tau)$, and their derivatives with respect to T are continuous in T and τ and $K(T, T) = 0$, $K_0(T, T) = -1$, it follows that

$$\delta_T'(T,f,c)=\int_0^T [K_0'(T,\tau)+c(\tau)K'(T,\tau)]\,f'(\tau)\,d\tau-f'(T)$$

if $f'(\tau)$ is continuous in T. If $f'(\tau)$ has a discontinuity at T, then for the derivatives from left and right, respectively, the second term of the formula is equal to $f'(T_{-0})$ and $f'(T_{+0})$, respectively. The derivatives of $K_0(T, \tau)$ and $K(T, \tau)$ are calculated with respect to the argument T. Since $f'(t)$ is not known in advance and merely satisfies the condition (3), it follows that

$$\sup_f|\delta(T,f,c)|=m\int_0^T |K_0(T,\tau)+c(\tau)K(T,\tau)|\,d\tau$$

$$\sup_f |\delta_T'(T, f, c)| = m\int_0^T |K_0'(T, \tau) + c(\tau) K'(T, \tau)| d\tau + m$$

Therefore

$$\inf_c \sup_f \delta|_T'(T, f, c)|, \qquad \inf_c m\int_0^T |K_0'(T, \tau) + c(\tau) K'(T, \tau)| d\tau$$

will be true for the same function $c_{min}(\tau)$.

It is shown in [2] that (6) is true for the function $c^{\circ}(\tau)$:

$$c^{\circ}(\tau) = -\frac{K_0(T, \tau)}{K(T, \tau)} \quad \text{for} \quad \left|\frac{K_0(T, \tau)}{K(T, \tau)}\right| < M$$
$$c^{\circ}(\tau) = -M \operatorname{sign} \frac{K_0(T, \tau)}{K(T, \tau)} \quad \text{for} \quad \left|\frac{K_0(T, \tau)}{K(T, \tau)}\right| \geqslant M \tag{8}$$

We set

$$K_0(T, \tau) + c(\tau) K(T, \tau) = N(\tau) + \varphi(\tau) K(\tau)$$
$$K_0'(T, \tau) + c(\tau) K'(T, \tau) = R(\tau) + \varphi(\tau) G(\tau) \tag{9}$$

where

$$N(\tau) = K_0(T, \tau) + c^{\circ}(\tau) K(T, \tau), \qquad R(\tau) = K_0'(T, \tau) + c^{\circ}(\tau) K'(T, \tau)$$
$$\varphi(\tau) = c(\tau) - c^{\circ}(\tau), \qquad K(\tau) = K(T, \tau), \qquad G(\tau) = K'(T, \tau) \tag{10}$$
$$a(\tau) = -M - c^{\circ}(\tau) \leqslant \varphi(\tau) \leqslant M - c^{\circ}(\tau) = b(\tau) \tag{11}$$

The above problem is thus equivalent to the following:

On the set D of functions $\varphi(\tau)$ satisfying (11) and the condition

$$Q(\varphi) = m\int_0^T |N(\tau) + \varphi(\tau) K(\tau)| d\tau < A \tag{12}$$

required to find the function $\varphi_{min}(\tau)$ for which

$$\inf_\varphi E(\varphi) = \inf_\varphi m\int_0^T |R(\tau) + \varphi(\tau) G(\tau)| d\tau \tag{13}$$

We set

$$\varphi^{\circ}(\tau) = a(\tau) \quad \text{for} \quad -\frac{R(\tau)}{G(\tau)} \leqslant a(\tau), \qquad \varphi^{\circ}(\tau) = b(\tau) \quad \text{for} \quad -\frac{R(\tau)}{G(\tau)} \geqslant b(\tau)$$
$$\varphi^{\circ}(\tau) = -\frac{R(\tau)}{G(\tau)} \quad \text{for} \quad a(\tau) < -\frac{R(\tau)}{G(\tau)} < b(\tau)$$

Let $\sigma(y)$ be a subset of $[0, T]$ such that if $\tau \in \sigma(y)$, then

$$\frac{G(\tau)}{K(\tau)} \geqslant y, \qquad y \in [B^-, B^+], \qquad B^- = \inf_\tau \frac{G(\tau)}{K(\tau)}, \qquad B^+ = \sup_\tau \frac{G(\tau)}{K(\tau)}$$

We now consider the function $\psi(\tau, y)$, dependent on the parameter y,

$$\psi(\tau, y) = \varphi^\circ(\tau) \quad \text{for } \tau \in \sigma(y), \qquad \psi(\tau, y) = 0 \quad \text{for } \tau \in \lambda(y) = [0, T] \setminus \sigma(y)$$

and the function

$$\Phi(y) = m \int_0^T |N(\tau) + \psi(\tau, y) K(\tau)| \, d\tau$$

Let us now note that for any arbitrary constant d the equation

$$\frac{G(\tau)}{K(\tau)} = d$$

can have only a finite number of zeros in the interval $[0, T]$. In fact

$$G(\tau) = \sum_{r=1}^n y_r'(T) Z_r(\tau), \qquad K(\tau) = \sum_{r=1}^n y_r(T) Z_r(\tau)$$

$$Z^\circ(\tau) \equiv G(\tau) - dK(\tau) = \sum_{r=1}^n [y_r'(T) - dy_r(T)] Z_r(\tau)$$

In the last equality at least one of the coefficients $y_r'(T) - dy_r(T)$ is different from zero, since otherwise the Wronskian would be equal to zero. Therefore, $Z^\circ(\tau)$ is a nontrivial solution of Equations (7). But this solution can have only a finite number of zeros in the interval $[0, T]$. The function $\Phi(y)$ is continuous and decreases monotonically as y varies from B^- to B^+.

Indeed, if $y_2 > y_1$ then $\sigma(y_2) \subset \sigma(y_1)$. Therefore $|\psi(\tau, y_1)| \geqslant |\psi(\tau, y_2)|$.

It follows from (8) to (11) that if $N(\tau)\psi(\tau, y)K(\tau) \neq 0$, then

$$\operatorname{sign}[N(\tau) + \psi(\tau, y) K(\tau)] = \operatorname{sign} N(\tau) = \operatorname{sign} \psi(\tau, y) K(\tau)$$

These equations may be obtained as follows: We have

$$\operatorname{sign} N = \operatorname{sign} K_0, \qquad c^\circ = -M \operatorname{sign} \frac{K_0}{K}, \qquad |K_0| > M|K|, \qquad \psi = \varphi^\circ$$

and therefore

$$a = M\left(-1 + \operatorname{sign} \frac{K_0}{K}\right) \leqslant \varphi^\circ \leqslant M\left(1 + \operatorname{sign} \frac{K_0}{K}\right) = b, \qquad -M \leqslant M \operatorname{sign} \frac{K_0}{K} - \varphi \leqslant M$$

Since

$$N + \psi K = K_0 - K\left(M \operatorname{sign}\frac{K_0}{K} - \varphi^\circ\right)$$

it follows that

$$\operatorname{sign}(N + \psi K) = \operatorname{sign} K_0 = \operatorname{sign} N$$

Moreover, $\operatorname{sign} \psi K = \operatorname{sign} \varphi^{\circ} K = \operatorname{sign} K_0 = \operatorname{sign} N$, since

$$-2M \leqslant \varphi^\circ < 0 \quad \text{if} \quad \operatorname{sign}\frac{K_0}{K} = -1, \qquad 0 < \varphi^\circ \leqslant 2M \quad \text{if} \quad \operatorname{sign}\frac{K_0}{K} = 1$$

Therefore $\Phi(y_1) \geqslant \Phi(y_2)$. The continuity of $\Phi(y)$ follows from the above noted property of $G(\tau)/K(\tau)$. Two cases are possible: either $\Phi(B^-) > A$ or $\Phi(B^-) \leqslant A$.

In the first case

$$\varphi_{\min}(\tau) = \psi(\tau, y_0) \equiv \psi^\circ(\tau) \tag{14}$$

where y_0 is the smallest root of the equation

$$\Phi(y) = A \tag{15}$$

Equation (15) has at least one root, since

$$\Phi(B^+) = m \int_0^T |N(\tau)| \, d\tau = A^\circ < A$$

It follows from the above that $\psi(\tau, y_i) \equiv \psi(\tau, y_j)$ if y_i and y_j satisfy (15).

In the second case

$$\varphi_{\min}(\tau) = \psi(\tau, B^-) = \varphi^\circ(\tau)$$

It is evident that $\psi^{\circ}(\tau)$ and $\varphi^{\circ}(\tau)$ belong to the set D.

We shall prove Equation (14). Let $\varphi(\tau)$ be an arbitrary function belonging to D.

Noting that for any τ belonging to $\sigma(y_0)$ we have

$$|\varphi| + |\psi^\circ| \geqslant \left|\frac{R}{G} + \varphi\right| - \left|\frac{R}{G} + \psi^\circ\right| = \left|\frac{R}{G} + \varphi\right| - \left|\frac{R}{G}\right| + |\psi^\circ| \geqslant |\psi^\circ| - |\varphi|$$

$$\left|\frac{R}{G} + \varphi\right| \geqslant \left|\frac{R}{G} + \psi^\circ\right|, \qquad \frac{G(\mu)}{K(\mu)} > \frac{G(\nu)}{K(\nu)} \quad \text{for } \mu \in \sigma(y_0),\ \nu \in \lambda(y_0)$$

$$\frac{A}{m} = \int_0^T |N + \psi^\circ K| \, d\tau = \int_0^T |N| \, d\tau + \int_{\sigma(y_0)} |\psi^\circ| |K| \, d\tau = \frac{A^\circ}{m} + \int_{\sigma(y_0)} |\psi^\circ| |K| \, d\tau$$

$$\frac{Q(\varphi)}{m} = \int_0^T |N + \varphi K|\, d\tau \geqslant \frac{A^\circ}{m} + \int_0^T |\varphi\|K|\, d\tau$$

it follows that

$$\int_0^T [\,|R + \varphi G| - |R + \psi^\circ G|\,]\, d\tau = \int_{\sigma(y_0)} |K| \left|\frac{G}{K}\right| \left[\left|\frac{R}{G} + \varphi\right| - \left|\frac{R}{G} + \psi^\circ\right|\right] d\tau +$$

$$+ \int_{\lambda(y_0)} |K| \left|\frac{G}{K}\right| \left[\left|\frac{R}{G} + \varphi\right| - \left|\frac{R}{G}\right|\right] d\tau = \left|\frac{G(\tau^*)}{K(\tau^*)}\right| \int_{\sigma(y_0)} |K| \left[\left|\frac{R}{G} + \varphi\right| - \right.$$

$$\left. - \left|\frac{R}{G} + \psi^\circ\right|\right] d\tau + \int_{\lambda(y_0)} |K| \left|\frac{G}{K}\right| \left[\left|\frac{R}{G} + \varphi\right| - \left|\frac{R}{G}\right|\right] d\tau \geqslant$$

$$\geqslant \left|\frac{G(\tau^*)}{K(\tau^*)}\right| \int_{\sigma(y_0)} |K| [\,|\psi^\circ| - |\varphi|\,]\, d\tau - \left|\frac{G(\tau^{**})}{K(\tau^{**})}\right| \int_{\lambda(y_0)} |K\|\varphi|\, d\tau =$$

$$= \left|\frac{G(\tau^*)}{K(\tau^*)}\right| \frac{(A - A^\circ)}{m} - \left|\frac{G(\tau^*)}{K(\tau^*)}\right| \int_{\sigma(y_0)} |K\|\varphi|\, d\tau -$$

$$- \left|\frac{G(\tau^{**})}{K(\tau^{**})}\right| \int_{\lambda(y_0)} |K\|\varphi|\, d\tau \geqslant \frac{1}{m} \left|\frac{G(\tau^*)}{K(\tau^*)}\right| [A - A^\circ - (Q(\varphi) - A^\circ)] \geqslant 0$$

It is thus shown that if $E(\varphi)$ is attained on $\varphi^{\circ}(\tau)$, that is, Equation (14) is valid. Formula (16) is proved in a similar manner.

Let us consider the computational side of the above problem. The determination of the function $K(T, \tau)$ reduces to the calculation of the fundamental systems of equations $y_1(t)$, $y_2(t)$, ..., $y_n(t)$ and $Z_1(t)$, $Z_2(t)$, ..., $Z_n(t)$. The methods of finding these solutions by means of high-speed digital computers or analog computers are well known. Thereafter the functions $N(\tau)$, $R(\tau)$, $\varphi^{\circ}(\tau)$ can easily be found. Since $\Phi(y)$ is a monotonic continuous function, a suitable method for solving Equations (15) to a predetermined degree of accuracy is the well-known numerical method of successive approximations called the rule of false position.

From the above property of the function $G(\tau)/K(\tau)$ it follows that for any given y the set $\sigma(y)$ consists of a finite number of intervals. The boundaries of these intervals are the roots of the equation

$$\frac{G(\tau)}{K(\tau)} = y$$

Finding the roots of this equation is equivalent to finding the zeros of the solution

$$z^\circ(\tau) = \sum_{r=1}^{n} [y_r'(T) - yy_r(T)] Z_r(\tau)$$

of Equations (7), which may be done by means of digital or analog computers. The calculation of the value of the function $\Phi(y)$ after finding the structure of the set $\sigma(y)$ reduces to the calculation of a finite number of intervals.

To illustrate the method of finding the function $\Phi(y)$, we shall consider an example.

Let the equation

$$y'' + y = f(t) + c(t) f'(t), \qquad y(0) = y'(0) = f(0) = 0, \quad |f'(t)| \leqslant m; \ |c(t)| \leqslant M$$

be given.

As is known, for this equation $K(T, \tau) = \sin(T - \tau)$. For simplicity, we set $T = 2\pi n$, where n is a positive integer. Then

$$K(T, \tau) = -\sin\tau, \qquad K_0(T, \tau) = \int_\tau^T K(T, u)\, du - 1 = -\cos(T - \tau) = -\cos\tau$$

$$K'(T, \tau) = \cos(T - \tau) = \cos\tau, \qquad K_0'(T, \tau) = \sin(T - \tau) = -\sin\tau$$

$$c^\circ(\tau) = -\cot\tau \quad \text{if} \ |\cot\tau| \leqslant M, \qquad c^\circ(\tau) = -M \operatorname{sign} \cot\tau \quad \text{if} \ |\cot\tau| \geqslant M$$

Using Formulas (10) and (11), we obtain

$$\begin{array}{lll} N(\tau) = 0, & R(\tau) = -\operatorname{cosech}\tau & (|\cot\tau| \leqslant M) \\ N(\tau) = -\cos\tau + M\sin\tau, & R(\tau) = -\sin\tau - M\cos\tau & (\cot\tau \geqslant M) \\ N(\tau) = -\cos\tau - M\sin\tau, & R(\tau) = -\sin\tau + M\cos\tau & (\cot\tau \leqslant -M) \end{array}$$

$$\begin{array}{lll} a(\tau) = -M + \cot\tau, & b(\tau) = M + \cot\tau & (|\cot\tau| \leqslant M) \\ a(\tau) = 0, & b(\tau) = 2M & (\cot\tau \geqslant M) \\ a(\tau) = -2M, & b(\tau) = 0 & (\cot\tau \leqslant -M) \end{array}$$

$$\frac{G(\tau)}{K(\tau)} = -\cot\tau, \qquad \begin{array}{lll} -\dfrac{R(\tau)}{G(\tau)} = \dfrac{1}{\sin\tau\cos\tau} = \dfrac{2}{\sin 2\tau} & & (|\cot\tau| \leqslant M) \\ -\dfrac{R(\tau)}{G(\tau)} = \dfrac{\sin\tau + M\cos\tau}{\cos\tau} = \tan\tau + M & & (\cot\tau \geqslant M) \\ -\dfrac{R(\tau)}{G(\tau)} = \dfrac{\sin\tau - M\cos\tau}{\cos\tau} = \tan\tau - M & & (\cot\tau \leqslant -M) \end{array}$$

Consequently, $\tau \in \sigma(y)$ if $-\cot\tau \geqslant y$. Let τ_y be a value of $\cot^{-1}(-y)$ belonging to $[0, \pi]$. Then $\sigma(y)$ consists of the intervals $[\tau_y, \pi]$, $[\tau_y + \pi + 2\pi]$, ..., $\tau_y + \pi(2n - 1)$, $2\pi n)$. We assume for the sake of definiteness that $M > 1$. Then from the determination of $\varphi^\circ(t)$ it follows that

$$\varphi^\circ(\tau) = \tan\tau + M \qquad (\cot\tau \geqslant M)$$
$$\varphi^\circ(\tau) = \tan\tau + \cot\tau \qquad (M \geqslant |\cot\tau| \geqslant M^{-1})$$
$$\varphi^\circ(\tau) = M + \cot\tau \qquad (M^{-1} \geqslant \cot\tau \geqslant 0)$$
$$\varphi^\circ(\tau) = -M + \cot\tau \qquad (-M^{-1} \leqslant \cot\tau < 0)$$
$$\varphi^\circ(\tau) = \tan\tau - M \qquad (\cot\tau < -M)$$

In the interval $[0, \pi]$

$$\psi(\tau, y) = \varphi^\circ(\tau) \quad \text{if} \quad \tau \in [\tau_y, \pi], \qquad \psi(\tau, y) = 0 \quad \text{if} \quad \tau \in [0, \tau_y]$$

Let

$$\cot\tau_0 = M, \qquad \cot\tau_1 = M^{-1} \qquad (\tau_0, \tau_1 \in [0, \pi])$$

Then for $y \leqslant -M$

$$\Phi(y) = 4\pi nm \left\{ \int_0^{\tau_y} |-\cos\tau + M\sin\tau|\, d\tau + \int_{\tau_y}^{\tau_0} \left| -\frac{1}{\cos\tau} \right| d\tau + \int_{\tau_0}^{\tau_1} \left| \frac{1}{\cos\tau} \right| d\tau + \right.$$
$$+ \int_{\tau_1}^{\pi/2} |(M + \cot\tau)(-\sin\tau)|\, d\tau + \int_{\pi/2}^{\pi-\tau_1} |(-M + \cot\tau)(-\sin\tau)|\, d\tau +$$
$$\left. + \int_{\pi-\tau_1}^{\pi-\tau_0} \left| -\frac{1}{\cos\tau} \right| d\tau + \int_{\pi-\tau_0}^{\pi} \left| -\frac{1}{\cos\tau} \right| d\tau \right\} =$$
$$= 4\pi nm \left\{ \frac{1 - My}{\sqrt{1 + y^2}} + 2 - M + \frac{1}{2} \ln \left[\frac{\sqrt{1 + y^2} - 1}{\sqrt{1 + y^2} + 1} \left(\frac{\sqrt{1 + M^2} + M}{\sqrt{1 + M^2} - M} \right)^2 \right] \right\}$$

Performing similar calculations, we find that

$$\Phi(y) = 4\pi nm \left\{ \sqrt{1 + M^2} + 2 - M + \frac{1}{2} \ln \left[\frac{\sqrt{1 + y^2} - 1}{\sqrt{1 + y^2} + 1} \left(\frac{\sqrt{1 + M^2} + M}{\sqrt{1 + M^2} - M} \right)^2 \right] \right\} \qquad (-M \leqslant y \leqslant -M^{-1})$$

$$\Phi(y) = 4\pi nm \left\{ \sqrt{1 + M^2} + 2 - M - \frac{My + 1}{\sqrt{1 + y^2}} + \frac{1}{2} \ln \frac{\sqrt{1 + M^2} + M}{\sqrt{1 + M^2} - M} \right\} \qquad (-M^{-1} \leqslant y \leqslant 0)$$

$$\Phi(y) = 4\pi nm \left\{ \sqrt{1 + M^2} - M + \frac{1 - My}{\sqrt{1 + y^2}} + \frac{1}{2} \ln \frac{\sqrt{1 + M^2} + M}{\sqrt{1 + M^2} - M} \right\} \qquad (0 \leqslant y \leqslant M^{-1})$$

$$\Phi(y) = 4\pi nm \left\{ \sqrt{1 + M^2} - M + \frac{1}{2} \ln \frac{\sqrt{1 + y^2} + 1}{\sqrt{1 + y^2} - 1} \right\} \qquad (M^{-1} \leqslant y \leqslant M)$$

$$\Phi(y) = 4\pi nm \left\{ 2\sqrt{1 + M^2} - M - \frac{1 + My}{\sqrt{1 + y^2}} + \frac{1}{2} \ln \frac{\sqrt{1 + y^2} + 1}{\sqrt{1 + y^2} - 1} \right\} \qquad (M \leqslant y < \infty)$$

We note that

$$\Phi(B^-) = \Phi(-\infty) = 4\pi nm\left[2 + \ln\frac{\sqrt{1+M^2}+M}{\sqrt{1+M^2}-M}\right]$$

$$\Phi(B^+) = \Phi(\infty) = 8\pi nm\,[\sqrt{1+M^2} - M]$$

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